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Similarity solution of thermal boundary layers for laminar narrow axisymmetric jets

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Abstract

In this work, the similarity equation describing the thermal boundary layers of laminar narrow axisymmetric jets is derived based on boundary layer assumptions. The equation is solved exactly. Some properties of the thermal jet are discussed. By introducing new-defined non-dimensional coordinates, the similarity solution results in a "universal" format. The results can also be applied in the boundary layer problem of species diffusion.

Keywords: Similarity solution; Narrow axisymmetric jet; Thermal boundary layer

1. Introduction

Schlichting (White, 1991; Schlichting and Gersten, 2000) solved the narrow axisymmetric jet problem based on boundary layer assumptions by introducing a new similarity variable. In applications, it is possible that the incoming jet flow has a different temperature from that of the quiescent fluid, which is the "free hot jet" problem. The thermal boundary layer solution of a narrow plane jet was given by Schlichting and Gersten (2000). However, the thermal boundary layer of a laminar narrow axisymmetric jet was not discussed by Schlichting. In the book of Pai (1954), which is a book specialized for jets, the thermal boundary layer was also not included. Kanury (1975) described a thermal boundary layer solution for a cylindrical jet in his book on combustion. In his work, he assumed that the Prandtl number of the fluid is one. Therefore, his thermal boundary layer is similar to a momentum boundary layer. Fujii (1963) presented the solution of a buoyancy jet for a point heat source. The thermal boundary layer of a free circular jet, however, is different from that of a buoyancy jet. The analytical solution of a buoyancy jet exists only when the Prandtl number $Pr = 2$ and $Pr = 1$. In the current work, the similarity equation of the thermal boundary layer for a narrow axisymmetric jet is derived and solved exactly. The derivation and results can also be applied in mass transfer problem of species diffusion by changing thermal diffusivity into species diffusivity and Prandtl number into Schmidt number for narrow axisymmetric jets.

2. Mathematical formulation

The describing equation of a narrow axisymmetric jet, neglecting the body force, based on boundary layer assumptions can be shown (White, 1991) to be

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{v}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2}$$

subject to the boundary conditions

$$v(x,0) = 0, \quad \frac{\partial u}{\partial r}(x,0) = 0 \quad \text{and} \quad u(x,\infty) = 0 \tag{3}$$

It is assumed that the inflow fluid temperature is different from the temperature, T_0, of quiescent fluid. Thus, by neglecting the dissipation term, the energy equation becomes

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Nomenclature

c_p	specific heat per unit mass	Pr	Prandtl number
d_i	incoming flow jet diameter	R	non-dimensional radial coordinate
r	radial coordinate	Re_i	incoming flow Reynolds number
u	axial fluid velocity	T	fluid temperature
u_i	incoming flow jet velocity	T_0	quiescent fluid temperature
v	radial velocity	T_i	incoming jet fluid temperature
x	axial coordinate	T_n	non-dimensional fluid temperature
A	integration constant of thermal boundary layer solution	$\widehat{T}$	residual temperature
B	coefficient of thermal jet radius	X	non-dimensional axial coordinate
C	integration constant of momentum boundary layer solution	*Greeks*	
F	dimensionless free stream function	α	thermal diffusivity
F'	first derivative of F	η	similarity variable
F''	second derivative of F	μ	dynamic viscosity
H	total enthalpy flux	ν	kinematic viscosity
J	total momentum flux	ρ	fluid density
		θ	similarity temperature function

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\frac{\nu}{Pr}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{4}$$

with boundary conditions

$$\frac{\partial T}{\partial r}(x,0)=0 \quad \text{and} \quad T(x,\infty)=T_0 \tag{5}$$

Defining a new temperature as $\widehat{T}=T-T_0$ yields the boundary conditions

$$\frac{\partial \widehat{T}}{\partial r}(x,0)=0 \quad \text{and} \quad \widehat{T}(x,\infty)=0 \tag{6}$$

The energy equation will remain the same by changing T into $\widehat{T}$. The following derivation employs the same similarity variables employed by White (1991). Since the analytical solution of the momentum boundary layer has been given, attention will be paid to the thermal boundary layer. The similarity variable is defined as $\eta=r/x$. Then we can define $u=\nu F'/r$ and obtain $v=(\nu/r)(\eta F'-F)$. The similarity equation of the momentum boundary layer becomes (White, 1991)

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(F''-\frac{F'}{\eta}\right)=\frac{1}{\eta^2}\left(FF'-\eta F'^2-\eta FF''\right) \tag{7}$$

The analytical solution of Eq. (7) is

$$F(\eta)=\frac{(C\eta)^2}{1+(C\eta/2)^2} \tag{8}$$

where $C=(3J/16\pi\rho\nu^2)^{1/2}$ and J is the given momentum flux.

Using the same similarity variable, we can define $\widehat{T}=\nu(\theta(\eta)/r)$. The energy equation simplifies to

$$-\frac{F'\theta}{\eta}-\frac{F\theta'}{\eta}+\frac{F\theta}{\eta^2}=\frac{1}{Pr}\left(\theta''-\frac{\theta'}{\eta}+\frac{\theta}{\eta^2}\right) \tag{9}$$

subject to boundary conditions

$$\theta'(0)=0 \quad \text{and} \quad \theta(\infty)=0 \tag{10}$$

Eq. (9) combining with boundary conditions can be integrated as following:

$$\theta(\eta)=A\mathrm{e}^{-\int_0^\eta((Pr\,F(\varsigma)-1)/\varsigma)\,\mathrm{d}\varsigma} \tag{11}$$

The coefficient A can be determined by the enthalpy flux in the cross-section for any x. We define the enthalpy flux as

$$H=\int_0^\infty \rho c_p[T(x,r)-T_0]u(x,r)2\pi r\,\mathrm{d}r \tag{12}$$

Plugging Eq. (8) into Eq. (11) yields

$$\theta(\eta)=\frac{A\eta}{\left[1+(C\eta/2)^2\right]^{2Pr}} \tag{13}$$

It is known (Schlichting and Gersten, 2000) that, for a laminar narrow plane jet, the relevant $\theta(\eta)$ equals $(f')^{Pr}$. From Eq. (13), it is seen that this is not the case for a narrow axisymmetric jet. Substituting Eq. (13) into Eq. (12) yields

$$H=\frac{8\pi A\rho c_p\nu^2}{2Pr+1} \tag{14}$$

From Eq. (14), the coefficient A can be determined as

$$A=\frac{H(2Pr+1)}{8\pi\rho c_p\nu^2}=A(Pr) \tag{15}$$

The temperature $\widehat{T}$ becomes

$$\widehat{T} = \frac{H(2Pr+1)}{8\pi\rho c_p v x}\left(1+\frac{C^2\eta^2}{4}\right)^{-2Pr} \quad (16)$$

The maximum temperature at the central line of the jet will become

$$\widehat{T}_{\max} = \frac{H(2Pr+1)}{8\pi c_p \mu x} \quad (17)$$

The radius of the axisymmetric thermal jet can be defined as

$$r_{\text{jet}} = r|_{\hat{T}/\hat{T}_{\max}=0.01} \quad (18)$$

Then plugging Eqs. (16) and (17) into Eq. (18) yields

$$r_{\text{jet}} = \frac{2x}{C}\left(100^{1/2Pr}-1\right) = B(Pr)\frac{2x}{C} \quad (19)$$

It is interesting that the radius of the thermal jet is proportional to the distance from the jet exit. The relationship of $B(Pr)$ to Prandtl number is depicted in Fig. 1. It is found from Fig. 1 that, at the same x location, the thermal jet radius will decrease with the increase of Prandtl number. If we denote the diameter of the long circular slot d_i, fluid velocity u_i, fluid temperature T_i as Kanury (1975), then we obtain

$$J = \rho\frac{\pi d_i^2}{4}u_i^2 \quad \text{and} \quad H = \rho c_p \frac{\pi d_i^2}{4}(T_i - T_0)u_i$$

A non-dimensional temperature can be defined as

$$T_n = \frac{T-T_0}{T_i-T_0} = \frac{\widehat{T}}{T_i-T_0}$$

Therefore,

$$T_n = \frac{2Pr+1}{32}\frac{d_i Re_i}{x}\left(1+\frac{C^2\eta^2}{4}\right)^{-2Pr} \quad (20)$$

where $Re_i = d_i u_i/v$ is the incoming flow Reynolds number. If we assume a unit Prandtl number, Eq. (20) becomes

$$T_n = \frac{3}{32}\frac{d_i Re_i}{x}\left(1+\frac{C^2\eta^2}{4}\right)^{-2} \quad (21)$$

which is the solution of the thermal boundary layer for a laminar cylindrical jet on the basis of analogy to Schlichting's momentum boundary layer (Kanury, 1975). From the above derivation, it is known that $C = \frac{\sqrt{3}}{8}Re_i$, and plugging C and η yields the equation of the non-dimensional temperature

$$T_n = \frac{2Pr+1}{32}\frac{d_i Re_i}{x}\left[1+\frac{3}{256}\left(\frac{d_i Re_i}{x}\right)^2\left(\frac{r}{d_i}\right)^2\right]^{-2Pr} \quad (22)$$

After defining non-dimensional coordinates $X = x/d_i Re_i$ and $R = r/d_i$, we obtain

$$T_n = \frac{2Pr+1}{32}X^{-1}\left(1+\frac{3}{256}X^{-2}R^2\right)^{-2Pr} \quad (23)$$

The isothermal curves for Prandtl number of 0.7 are shown in Fig. 2. Because of the influence of Reynolds number, the X coordinate is greatly shrunk. The constant non-dimensional temperature curves for different Prandtl numbers are shown in Fig. 3. It is seen from the plot that the peak value of the non-dimensional R coordinate varies with Prandtl number. R_{peak} reduces with increasing of Prandtl number when $Pr \leqslant Pr_c$, where Pr_c is a critical Prandtl number of the fluid, while it increases with increasing of Prandtl number when $Pr \geqslant Pr_c$. For this case, say $T_n = 0.4$, the minimum R_{peak} happens at $Pr_c \approx 0.8$. The relationship between R_{peak} and Pr is

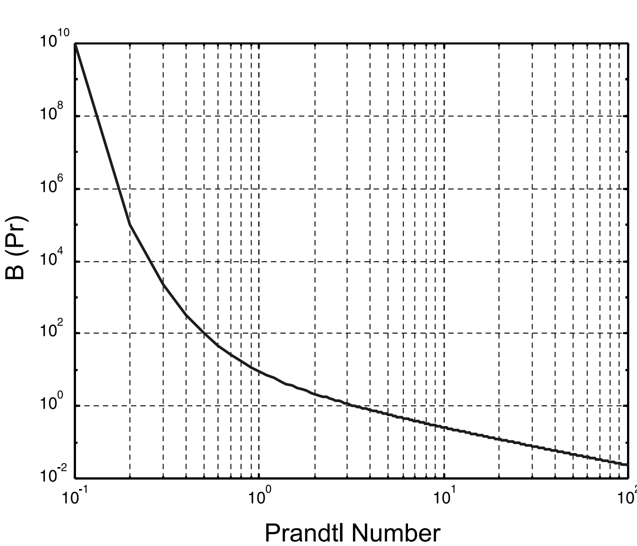

Fig. 1. The relationship of $B(Pr)$ to Prandtl number.

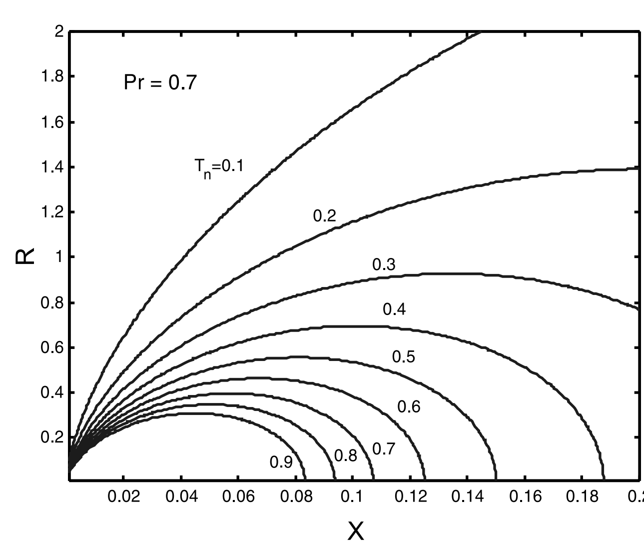

Fig. 2. Non-dimensional isothermal curves for $Pr = 0.7$.

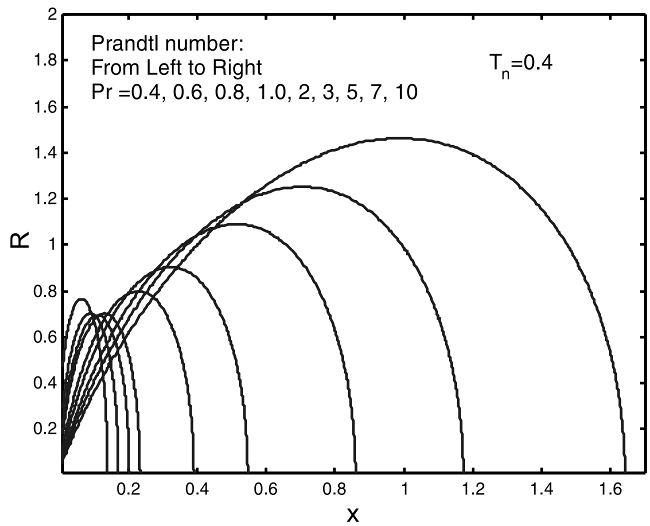

Fig. 3. Plots of constant non-dimensional temperature for different Prandtl numbers.

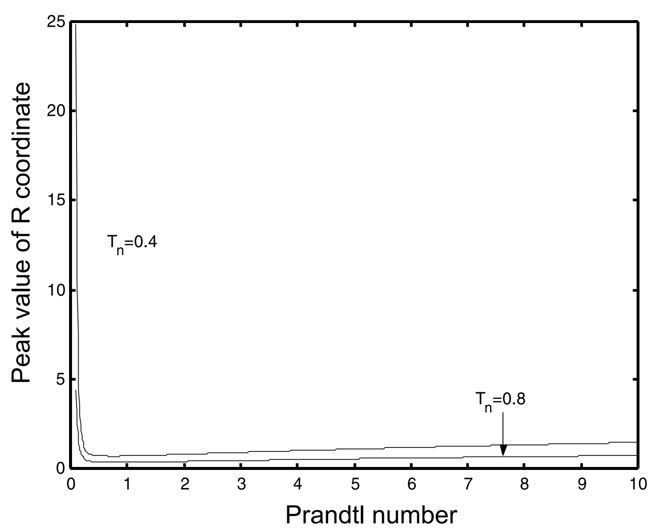

Fig. 4. The relationship between R_{peak} and Prandtl number for different non-dimensional temperatures.

shown in Fig. 4. Two non-dimensional temperatures are depicted in it. It is found that the above-mentioned relation is explained by the curves. The minimum R_{peak} happens for critical Prandtl number close to one.

3. Conclusion

In the current work, the thermal boundary layer of a laminar narrow axisymmetric jet is analyzed and the similarity equation describing the problem is derived and solved exactly. Other useful quantities relevant to thermal axisymmetric jets are also discussed. By defining relevant non-dimensional coordinates, the similarity solution is given in a "universal" format.

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